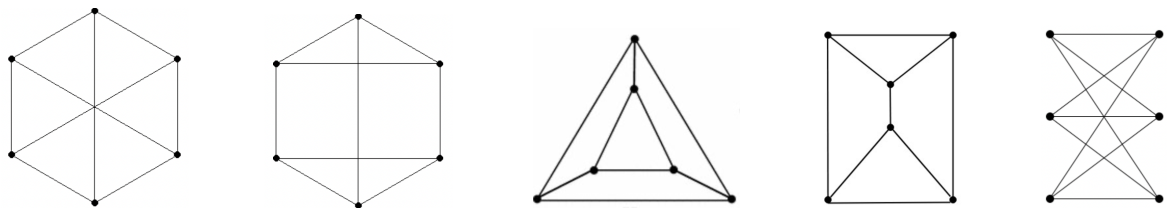
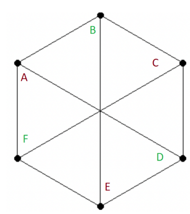
Problem Set 6 (Solutions by F. Gotti and D. Kliuev)

Problem 1 *Recall that two graphs G_1 and G_2 are called isomorphic if there exists a bijective function $f\colon V(G_1) \to V(G_2)$, called an isomorphism, such that $uv \in E(G_1)$ if and only if $f(u)f(v) \in E(G_2)$. Which of the following graphs are isomorphic? Justify your answer.*



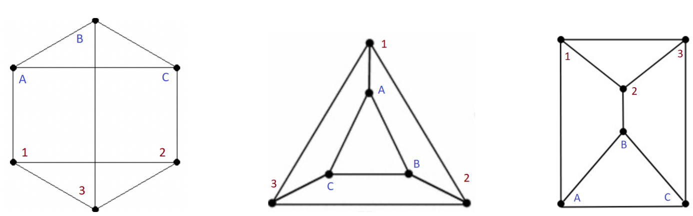
Solution. Let us first observe that isomorphisms preserve chromatic number (verify this!). Formally, if $f\colon V(G) \to V(G')$ is an isomorphism between graphs G and G' and $c\colon V(G') \to \mathbb{N}$ is a proper coloring of G', then $c \circ f\colon V(G) \to \mathbb{N}$ is a proper coloring of G. Similarly, if c is a proper coloring of G, then $c \circ f^{-1}$ is a proper coloring of G'.

Let G_1, G_2, G_3, G_4, and G_5 denote the graphs in the picture (from left to right). We note that G_1 is isomorphic to $K_{3,3}$, that is, it is isomorphic to G_5. It contains all edges between the set of red vertices $\{A, C, E\}$ and the set of green vertices $\{B, D, F\}$:



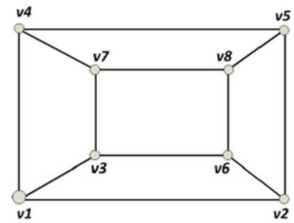
In particular, $\chi(G_1) = \chi(G_5) = 2$. Now observe that that graphs G_2, G_3, and G_4 all contain a triangle. Hence $\chi(G_i) > 2$ for $i \in [\![2,4]\!]$, and so none of the graphs G_2, G_3, and G_4 is isomorphic to G_1 (or G_5).

Finally, we claim that the graphs G_2, G_3, and G_4 are isomorphic to each other. Indeed, they all consist of two triangles ABC, 123 and edges $1A$, $2B$, $3C$, as illustrated in the following picture.



□

Problem 2 *Let G be a graph. If $f\colon V(G) \to V(G)$ is an isomorphism of graphs, then we call f an* ***automorphism*** *of G. How many automorphisms does the following graph have? Justify your answer.*



Solution. Let f be an automorphism of G. Since the vertices v_2, v_3, v_4 are all neighbors of v_1, $f(v_2)$, the vertices $f(v_3)$, $f(v_4)$ must be distinct neighbors of $f(v_1)$.

Suppose that f, g are automorphisms of G such that $f(v_i) = g(v_i)$ for every $i \in [4]$. Then $h := g^{-1} \circ f$ is an automorphism of G such that $h(v_i) = v_i$ for $i \in [4]$. Note that the vertices v_2 and v_4 have two common neighbors, namely, v_1 and v_5. Since h is bijection we have $h(v_5) = v_5$. Reasoning similarly, we can conclude that $h(v_6) = v_6$ and $h(v_7) = v_7$. Since h is bijection, $h(v_8) = v_8$. Thus, it follows that any automorphism of G is uniquely determined by the images of the vertices v_1, v_2, v_3, and v_4.

Let a, b, c, d be distinct vertices of G such that b, c, d are neighbors of a. We claim that there exists an automorphism f of G such that $f(v_1) = a$, $f(v_2) = b$, $f(v_3) = c$, $f(v_4) = d$. In order to prove this claim, we first note that G is isomorphic to the graph consisting of all the vertices and edges of a cube. It follows that any isometry of a cube gives an automorphism of G. We will think of G as the 1-skeleton (i.e., the graph of all vertices and edges) of a cube, and we will construct an isometry f of the Euclidean plane with the required properties. First we apply a translation by vector $\overrightarrow{v_1a}$. After that we apply an orthogonal transformation that fixes a and sends the vectors $\overrightarrow{v_1v_2}, \overrightarrow{v_1v_2}$, and $\overrightarrow{v_1v_2}$ to the vectors $\overrightarrow{ab}, \overrightarrow{ac}$, and $\overrightarrow{ad}$, respectively. Such transformation exists because $\{\overrightarrow{v_1v_2}, \overrightarrow{v_1v_2}, \overrightarrow{v_1v_2}\}$ and $\{\overrightarrow{ab}, \overrightarrow{ac}, \overrightarrow{ad}\}$ are orthogonal basis of $\mathbb{R}^3$. Using $\overrightarrow{v_2v_5} = \overrightarrow{v_1v_4}$ and similar equations, we deduce that f sends G to G. We note that f sends v_1, v_2, v_3, v_4 to a, b, c, d, as required.

As a result, we conclude that automorphisms of G are in one-to-one correspondence with quadruples (a, b, c, d) of distinct vertices of G such that b, c, d are neighbors of a. There are 8 ways to choose a and 3! ways to choose b, c, d, giving a total of 48 automorphisms. □

Problem 3 *Let G be a simple graph. We say that $e \in E(G)$ is a* bridge *if the graph $(V(G), E(G) \setminus \{e\})$ has more connected components than G. Let G be a bipartite k-regular graph for $k \geq 2$. Prove that G has no bridge.*

Solution. Suppose, by way of contradiction, that G contains a bridge e. Let W be a connected component that contains e. Without loss of generality, we may assume that $G = W$, which means that G is connected. Let $G = G_1 \cup G_2$, where all edges go between G_1 and G_2.

Let U and V be connected components of the graph we obtain from G after dropping e. Now set $U_i := U \cap G_i$ and $V_i := V \cap G_i$ for each $i \in [2]$. The graphs U and V are bipartite with parts (U_1, U_2) and (V_1, V_2), respectively. Since G is connected e must connect a vertex in U with a vertex in V. We can assume, without loss of generality, that e connects a vertex in U_1 with a vertex in V_2.

For each vertex $v \in U$, let $d_G(v)$ and $d_U(v)$ denote the degrees of v in G and U, respectively. For all vertices $v \in U$ except one, the equality $d_G(v) = d_U(v)$ holds. Letting S be the set of edges between U_1 and U_2, we obtain

$$k|U_2| = \sum_{v \in U_2} d_G(v) = \sum_{v \in U_2} d_U(v) = |S| = \sum_{v \in U_1} d_U(v) = \sum_{v \in U_1} d_G(v) - 1 = k|U_1| - 1.$$

However, this implies that 1 is divisible by k, contradicting the fact that $k \geq 2$. Thus, we conclude that G has no bridges. □

Problem 4 *For every $n \in \mathbb{N}$ with $n \geq 3$, find the chromatic polynomial of C_n, the cycle graph on $[n]$.*

Solution. For each $n \geq 3$, let $q_n(x)$ be the chromatic polynomial of C_n. For convenience, we allow $n = 2$ and define C_2 in this case to be a graph on two vertices consisting of one edge. We have seen before that, for every simple graph G, the following identity holds:

$$p_G(x) = p_{G \setminus e}(x) - p_{G/e}(x),$$

where $p_G(x)$, $p_{G \setminus e}(x)$, and $p_{G/e}(x)$ are the chromatic polynomials of G, its deletion $G \setminus e$, and its contraction G/e, respectively. Observe that for each $n \geq 3$, if $G = C_n$ and $e \in E(G)$, then $G/e = C_{n-1}$ and $G \setminus e = P_n$, a path of length n. Accordingly,

$$q_n(x) = p_{P_n}(x) - q_{n-1}(x).$$

We have seen in lectures that $p_{P_n}(x) = x(x-1)^{n-1}$. Let us proceed by induction on n to argue that

$$q_n(x) = (x-1)^n + (-1)^n(x-1)$$

for every $n \geq 2$. When $n = 2$, we see that $q_2(x) = P_{P_2}(x) = x(x-1) = (x-1)^2 + (x-1)$. The inductive step from n to $n+1$ goes as follows:

$$\begin{aligned} q_{n+1}(x) &= p_{P_{n+1}}(x) - q_n(x) \\ &= x(x-1)^n - (x-1)^n - (-1)^n(x-1) \\ &= (x-1)^{n+1} + (-1)^{n+1}(x-1). \end{aligned}$$

Hence $p_{C_n}(x) = q_n(x) = (x-1)^{n+1} + (-1)^{n+1}(x-1)$. □

Problem 5 *For $n \in \mathbb{N}$, prove that the chromatic polynomial of the complete bipartite graph $K_{n,2}$ is $x(x-1)^n + x(x-1)(x-2)^n$.*

Solution. Let $G = K_{n,2}$. It suffices to find a polynomial expression for $p_G(k)$ for any positive integer k with $k \ge 2$. Fix $k \in \mathbb{N}$ with $k \ge 2$, and let us find the number of proper k-coloring of $K_{n,2}$. Let $\{u, v\}$ be a size-2 part of the bipartite graph $K_{n,2}$. We split the rest of the proof into the following two cases.

Case 1: u and v have the same color. In this case, there are k colors to choose for u and v, and then there are $k-1$ colors to choose for each of the remaining vertices. This gives a total of $k(k-1)^n$ proper k-coloring in this first case.

Case 2: u and v have different colors. In this case, there are k colors to choose for u, there are $k-1$ colors to choose for v, and then there are $k-2$ colors to choose for each of the remaining vertices. This gives a total of $k(k-1)(k-2)^n$ proper k-coloring in this second case.

Hence we conclude that $p_G(k) = k(k-1)^n + k(k-1)(k-2)^n$, which give the desired chromatic polynomial for $K_{n,2}$. □

Problem 6 *Let G be a simple connected k-regular graph (with $k \ge 3$) that is neither an odd cycle nor a complete graph, and assume that G has no cut-vertices. Prove that if the subgraph $G \setminus \{v\}$ of G contains a cut-vertex for some $v \in V(G)$, then $\chi(G) \le \Delta(G)$.*

Solution. We will prove Brooks' theorem by induction on $|G|$. The base cases $|G| \in [2]$ are clear. The induction step was already proved in the lecture notes for all cases except when G satisfies the conditions in the statement of this problem. Hence it suffices to prove the statement of this problem.

Let u be a cut-vertex in graph $G' := G \setminus \{v\}$. Let H be the graph obtained from G' be removing u (along with the edges incident to u in G'), and let $H_1, \dots, H_\ell$ be the connected components of H. The fact that u is a cut-vertex of G' ensures that $\ell \ge 2$. Since G has no cut-vertices, $G \setminus \{u\}$ is connected. As there are no edges between H_i and H_j when $i \ne j$, the only possible edges between H_i and $V(G) \setminus (V(H_i) \cup \{u\})$ must connect some vertices of H_i to v. Since $G \setminus \{u\}$ is connected we deduce that, for each $i \in [\ell]$, there exists an edge connecting v to some of the vertices of H_i.

Now for every $i \in [\ell]$, let H_i' be the induced subgraph of G on the set of vertices $V(H_i) \cup \{u, v\}$. We note that $E(G) = E(H_1') \cup \cdots \cup E(H_\ell')$. Hence it is enough to color $H_1', \dots, H_\ell'$ such that we use the same color for u and we use the same color for v.

As for each $i \in [\ell]$, there is an edge of G connecting u to some vertex in H_i, the degree of u in H_i' is at most $k - \ell + 1$. The same statement holds for v. For each $i \in [\ell]$, set $K_i := H_i'$ if u and v are adjacent and set $K_i := (V(H_i'), E(H_i') \cup \{uv\})$ otherwise. The degree of u, v in K_i is at most $k + 2 - l \le k$.

Any vertex of H_i has degree $k \ge 3$ in K_i, and so K_i cannot be an odd cycle. Thus, if none of $K_1, \dots, K_\ell$ is a complete graph on $k+1$ vertices, then we can use Brooks' theorem on $K_1, \dots, K_\ell$ to obtain proper k-colorings of the graphs $H_1', \dots, H_\ell'$ all satisfying that u, v have different colors. After relabeling colors, one can assume that u has color 1 and v has color 2 in each of these proper k-colorings. This gives a proper coloring of G.

Finally, assume that $K_1, \dots, K_\ell$ are complete graphs on $k+1$ vertices. In particular, H_1 is a complete graph on $k-1$ vertices and there is an edge from both u and v to each vertex in H_1. Hence there is only one edge from u to $V(G) \setminus V(H_1')$, and the same statement holds for v. Since the degrees of u and v in K_1 is at most $k - \ell + 2$, the equality $\ell = 2$ must hold. We use now Brooks' theorem for H_2 to obtain a proper k-coloring of H_2. Now there are only two edges between $\{u, v\}$ and H_2. Hence there are at most two colors that we cannot use for u or v. Since $k \ge 3$, we can choose a third color and use it for both u and v. After that, we color H_1 using the $k-1$ colors that we did not use for $\{u, v\}$. This gives a proper k-coloring of G. □

Problem 7 *Is it possible to subdivide a square into finitely many concave quadrilateral?*

Solution. Assume, towards a contradiction, that we have subdivided a given square S into n concave quadrilaterals. Now consider this subdivision as a planar graph G with V vertices, E edges, and F faces. It is clear that the n concave angles of the concave quadrilaterals determine n vertices of G that are contained in the interior of the given square. Thus, $V \ge n + 4$ (as the corners of S are also vertices of G). On the other hand, G has $n+1$ faces, namely, the n concave quadrilaterals and the complement of the given square, which is the unbounded face. Since the boundary of each face consists of 4 edges, the fact that every edge is in the boundary of exactly two faces guarantees that $E = 4F/2 = 2(n+1)$. It follows now by the Euler's formula that $V = E - F + 2 = n + 3$, contradicting that $V \ge n + 4$. □

Problem 8 *Let P be a convex polyhedron with triangular faces. Suppose that the edges of P are oriented. A **singularity** of P is a face whose edges form an oriented cycle or a vertex v with indeg$(v) \cdot$ outdeg$(v) = 0$. Prove that P has at least two singularities.*

Solution. Let G be the directed graph whose vertices and arrows are the vertices and oriented edges of P, respectively. Let V, E, and F be the number of vertices, arrows, and faces of G, respectively. First, we find expressions for both E and F in terms of V.

Since the boundary of every face of G consists of 3 arrows, and every arrow belongs to the boundary of exactly two faces of G, it follows that $E = 3F/2$, and so

$$V = E - F + 2 = \frac{3F}{2} - F + 2 = \frac{F}{2} + 2$$

by virtue of the Euler's formula. Therefore

$$F = 2V - 4. \tag{1}$$

Now let N denote the number of oriented paths of length 2 in G. Observe that every vertex that is a singularity is the middle vertex of at least two paths in G of length 2. Therefore, after letting V_s denote the number of vertices of G that are singularities, we see that

$$N \geq 2(V - V_s) = 2V - 2V_s. \tag{2}$$

In addition, observe that each face of G that is not a singularity contributes with 1 to N, while each face of G that is a singularity contributes with 3 to N. Thus, after letting F_s denote the number of faces of G that are singularities, we obtain that

$$F + 2F_s = (F - F_s) + 3F_s = N \geq 2V - 2V_s, \tag{3}$$

where the last inequality is (2). Now we can use (1) in the inequality (3) to obtain that $2(V_s + F_s) \geq 2V - F = 4$. Hence the number of singularities $V_s + F_s$ of P is at least 2, which concludes the proof. □